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グラフの支配集合間の相互変換可能性について

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概要

本稿では，ネットワーク上で動作するモバイルサーバたちによって継続的なサービスを提供するための新しい理論の枠組みを提案する．以下では，与えられたサービスをネットワーク上で正しく提供することのできるモバイルサーバの配置を，そのネットワークに対応するグラフの支配集合としてモデル化し，各モバイルサーバは，グラフの辺にそってのみ移動できるものとする．以下の定理が示される：n 頂点の木の族において継続的なサービスを提供するためには $\lceil (n+1)/2 \rceil$ 個のモバイルサーバが必要かつ十分であり，n 頂点のハミルトングラフの族において継続的なサービスを提供するためには，$\lceil (n+1)/3 \rceil$ 個のモバイルサーバが必要かつ十分である．

On Mutual Transferability among Dominating Sets in Graphs

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Abstract

In this paper, we propose a new framework to provide continuous services by “mobile servers” in interconnection networks. We model those mobile servers as a subset of host nodes, and assume that a node can receive the service if at least one adjacent node plays the role of a server (i.e., we assume that the service could not be routed via the interconnection network). The main results are summarized as follows: For the classes of trees with n vertices, $\lceil (n+1)/2 \rceil$ mobile servers are necessary and sufficient to realize continuous services by the mobile servers, and for the classes of Hamiltonian graphs with n vertices, $\lceil (n+1)/3 \rceil$ mobile servers are necessary and sufficient.

1 Introduction

In this paper, we propose a new framework to provide continuous services by “mobile servers” in interconnection networks. We model those mobile servers as a subset of host nodes, and assume that a node can receive the service if at least one adjacent node plays the role of a server (i.e., we assume that the service could not be routed via the interconnection network). In other words, to provide the same service to all the users, the set of servers must be a **dominating set** [3] for the given network. Under such an abstract model, we will consider the following theoretical problem: *Given two dominating configurations A and B, can we transfer configuration A to B by keeping continuous services to the users?* The definition of the term “continuous service” will be given later. The main results obtained in this paper could be summarized as follows: *For the classes of trees with n vertices, $\lceil (n+1)/2 \rceil$ mobile servers are necessary and sufficient to realize mutual transfers among dominating configurations, and for the classes of Hamiltonian graphs with n vertices, $\lceil (n+1)/3 \rceil$ mobile servers are necessary and sufficient.*

Readers should note that, to the authors' best knowledge, this is the first paper to investigate the mutual transferability among dominating configu-

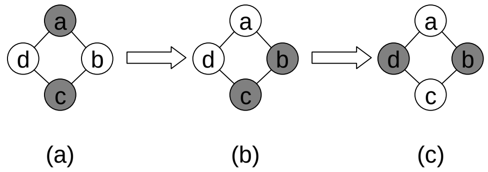


Figure 1: A sequence of single-step transfers among dominating configurations for a ring with four vertices (dominating vertices are painted gray).

rations by mobile servers, although the relation between domination and server assignment has been pointed out in the literature [1, 2, 3].

2 Preliminaries

Let $G = (V(G), E(G))$ be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. A dominating set for G is a subset U of $V(G)$ such that for any vertex $u \in V(G)$, either $u \in U$ or there exists a vertex $v \in U$ such that $\{u, v\} \in E(G)$. In this paper, by technical reasons, we assume that dominating set is a multiset; i.e., it can contain each vertex in $V(G)$ several times. Let $\mathcal{D}(G)$ denote an (infinite) set of all dominating (multi)sets for G. A dominating set is said to be minimal if the removal of any vertex from that violates the condition of domination (by definition, any minimal dominating set cannot be a multiset). The **domination number** $\gamma(G)$ of G is the size of a minimum dominating set for G, and the **upper domination number** $\Gamma(G)$ of G is the size of a minimal dominating set for G with a maximum cardinality [3].

For any $S_1, S_2 \in \mathcal{D}(G)$, we say that S_1 is *single-step transferable* to S_2, and denote it as $S_1 \to S_2$, if there are two vertices u and v in $V(G)$ such that $S_1 - \{u\} = S_2 - \{v\}$ and $\{u, v\} \in E(G)$. Note that a single-step transfer from S_1 to S_2 is realized by moving the "role of dominating vertex" from $u \in S_1$ to its neighbor $v \in S_2$, where each vertex can own more than one roles, since each dominating set is assumed to be a multiset. For example, in a ring network consisting of four vertices $\{a, b, c, d\}$, a dominating configuration with vertices $\{a, c\}$ is transferred to a dominating configuration with vertices $\{b, c\}$ in a single-step by moving the role of dominating vertex from a to its neighbor b (see Figure 1, for illustration). A transitive closure of the relation of single-step transferability naturally defines the notion of transferability, that will be denoted as $S_1 \xrightarrow{*} S_2$, in what follows. Note that every subset of vertices appearing in a transfer from S_1 to S_2 must be a dominating set for G. A set $\mathcal{D}' \subseteq \mathcal{D}(G)$ is said to be **mutually transferable** if it holds $S_1 \xrightarrow{*} S_2$ for any $S_1, S_2 \in \mathcal{D}'$, where a sequence of single-step transfers from S_1 to S_2 can contain a subset not in $\mathcal{D}'$, although all subsets in it must be an element in $\mathcal{D}(G)$.

3 Main Theorems

The first theorem gives a tight bound for the class of trees (proofs of all theorems will be given in the next section).

Theorem 1 (Trees) *For any tree T with n vertices, the set of dominating sets for T consisting of $k \geq \lfloor n/2 \rfloor$ vertices is mutually transferable, and there is a tree T with n vertices such that $\gamma(T) = \lfloor n/2 \rfloor$.*

Next, we provide a lower bound on the number of dominating vertices that is necessary to guarantee the mutual transferability among dominating sets, for all graphs contained in a class of Hamiltonian graphs with n vertices.

Theorem 2 (Lower Bound) *For any $r \geq 2$ and $n \geq 1$, there is a Hamiltonian r-regular graph G with more than n vertices such that the set of dominating sets for G with cardinality at least $\lceil (n+1)/3 \rceil - 1$ is not mutually transferable.*

It is worth noting that for any Hamiltonian graph G consisting of n vertices, $\gamma(G) \leq \lceil (n+1)/3 \rceil - 1$, since it contains a ring of size n as a subgraph. It is in contrast to the case of trees, since the theorem claims that there is a Hamiltonian r-regular graph G such that $\lceil (n+1)/3 \rceil - 1$ dominating vertices are not sufficient to guarantee the mutual transferability among dominating configurations, while $\lceil (n+1)/3 \rceil - 1$ vertices are sufficient to dominate it. By combining Theorem 2 with the following theorem, we could derive the tightness of $\lceil (n+1)/3 \rceil$ bound for the class of Hamiltonian graphs with n vertices.

Theorem 3 (Hamiltonian Graphs) *For any Hamiltonian graph G with n vertices, the set of dominating sets for G consisting of $k \geq \lceil (n+1)/3 \rceil$ vertices is mutually transferable.*

4 Sketch of Proofs

4.1 Theorem 1

Let T be a tree with at least two vertices. Let u be a leaf vertex in T and v be the unique neighbor of

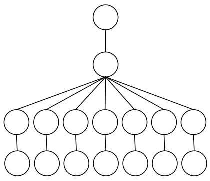
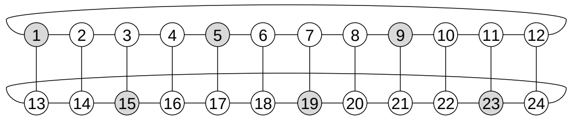
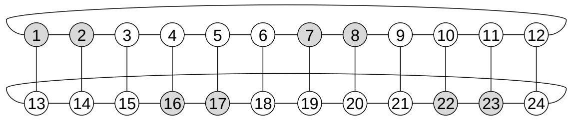


Figure 2: An example of tree consisting of 16 vertices with domination number eight.



(a) A minimum dominating set for G_1.



(b) A dominating set for G_1.

Figure 3: Two dominating sets for graph G_1 (dominating vertices are painted gray).

u. Any dominating set for T containing u is (single-step) transferable to a dominating set that contains v instead of u, and this transformation allows us to reduce the problem of dominating a tree with n vertices by a set with $k = \lfloor n/2 \rfloor$ dominating vertices to the problem of dominating a tree with at most $n-2$ vertices by a set with $k-1 = \lfloor (n-2)/2 \rfloor$ dominating vertices.

By repeatedly applying this operation, we will have a situation in which either: 1) a tree consisting of at most three vertices is dominated by one vertex, or 2) a tree consisting of n' vertices is dominated by (at least) n' vertices, and in each of the cases, dominating configurations corresponding to the case are trivially mutually transferable. Note that a sequence of such reduction steps is characterized by a sequence of vertices, that are leaves in the corresponding reduced trees. Since any dominating configuration with k vertices can be transferred to such normalized configurations with the same sequence of vertices, we can conclude that the set of dominating sets for T with k vertices is mutually transferable via those normalized configurations. An example of tree requiring $\lfloor n/2 \rfloor$ dominating vertices is illustrated in Figure 2. Hence the theorem follows.

4.2 Theorem 2

The given claim immediately holds for $r = 2$ since in ring networks consisting of $3m$ vertices, two dominating sets with $\lceil (3m+1)/3 \rceil - 1 = m$ vertices are not mutually transferable if $m \geq 2$.

For $r = 3$, we may consider the following graph $G_1 = (V_1, E_1)$ consisting of 24 vertices, where

$$\begin{aligned} V_1 &\stackrel{\text{def}}{=} \{1, 2, \ldots, 24\}, \quad \text{and} \\ E_1 &\stackrel{\text{def}}{=} \{(i, i+1) \mid 1 \leq i \leq 11, 13 \leq i \leq 23\} \\ &\quad \cup \{(12, 1), (24, 13)\} \\ &\quad \cup \{(i, i+12) \mid 1 \leq i \leq 12\}. \end{aligned}$$

Note that G_1 is Hamiltonian, cubic, and the domination number of G_1 is six (e.g., $\{1, 5, 9, 15, 19, 23\}$ is a minimum dominating set for G_1; see Figure 3

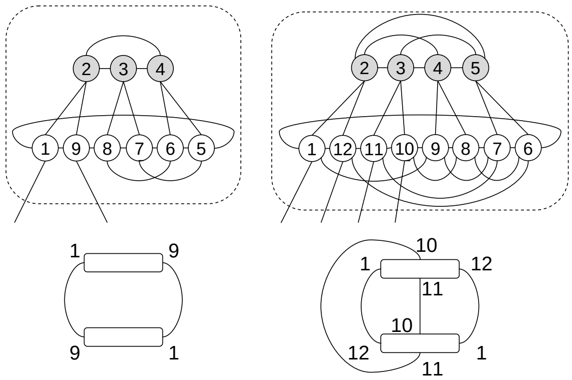


Figure 4: Extension to larger r's (the upper figures represent the basic component and the lower figures represent how to connect of those components; the label associated with connecting edges is the label of terminal vertices in each component).

(a) for illustration). Now let us consider a subset of vertices $S = \{1, 2, 7, 8, 16, 17, 22, 23\}$. It is obvious that S is a dominating set for G_1 and *any* vertex in S cannot move the role of dominating vertex to its neighbor without violating the condition of domination, since each vertex in S "privately" dominates two vertices. For example, as is shown in Figure 3 (b), in the dominating set S for G_1, vertex 2 dominates vertices 3 and 14 and those two vertices are not dominated by the other vertices. Thus, in order to realize a mutual transfer among dominating sets, nine ($= \lceil (24+1)/3 \rceil$) vertices are necessary. The above construction can be directly extended to larger cubic graphs consisting of $24x$ vertices for all $x \geq 1$; i.e., we can show that $8x + 1$ ($= \lceil (24x+1)/3 \rceil$) dominating vertices are necessary to guarantee the mutual transferability among dominating sets.

An extension to larger r's can be easily realized as well, i.e., we can show that there is a Hamiltonian

r-regular graph consisting of $3(r-1)x$ vertices for $x \geq 2$, such that $(r-1)x+1$ dominating vertices are necessary to guarantee the mutual transferability (see Figure 4 for illustration). Hence the theorem follows.

4.3 Theorem 3

The proof of Theorem 3 consists of two parts. In the first part, we show that the claim holds if we restrict the underlying Hamiltonian graph to rings (note that a ring is a "simplest" Hamiltonian graph). The second part gives a transfer of a dominating set for a Hamiltonian graph to a dominating set for a Hamiltonian cycle contained in it.

4.3.1 Proof for Rings

A ring network consisting of n vertices, denoted as $R_n = (V(R_n), E(R_n))$, is formally defined as follows: $V(R_n) \stackrel{\text{def}}{=} \{0, 1, \ldots, n-1\}$ and $E(R_n) \stackrel{\text{def}}{=} \{(i, i+1) \mid 0 \leq i \leq n-2\} \cup \{(n-1, 0)\}$. In what follows, we denote $i \pmod{n}$ by i for brevity, and represent a dominating set for R_n by a binary sequence of length n; e.g., sequence `010111` represents a dominating set $\{1, 3, 4, 5\}$ for R_6.

Lemma 1 (Rings) *For ring R_n with n vertices, the set of dominating sets for R_n consisting of $k \geq \lceil (n+1)/3 \rceil$ vertices is mutually transferable.*

Proof. Since the given claim obviously holds for $n \leq 3$, we may assume $n \geq 4$, without loss of generality. In the following proof, we examine the following three cases separately, where Case 2 will be reduced to Case 1, and Case 3 will be reduced to Case 2.

Case 1: When $n = 3m+2$ for some integer $m \geq 1$.

Case 2: When $n = 3m+1$ for some integer $m \geq 1$.

Case 3: When $n = 3m$ for some integer $m \geq 1$.

Case 1: In R_n, any vertex can dominate at most three vertices including itself. Since $n = 3m+2$ and $k = \lceil (n+1)/3 \rceil = m+1$, in any dominating set with k vertices, two vertices dominate five vertices, and the remaining $k-2$ vertices dominate $3(k-2) = 3(m-1)$ vertices. Thus, any dominating set in this case can be characterized by the vertex commonly dominated by two vertices. Let S_i denote the dominating set containing vertices $i-1$ and $i+1$; i.e., i is the vertex dominated by two vertices. By moving the role of dominating vertex from $i+1$ to $i+2$, we realize a single-step transfer from S_i to S_{i+3}, and by repeatedly applying the same operation, we can reach dominating set S_{i+3x} for any $x \geq 1$. Since $n \pmod{3} \not\equiv 0$, there exists an integer satisfying $j \equiv i + 3x \pmod{n}$ for any $0 \leq j \leq n-1$. Hence, the claim follows.

$$0110 \rightarrow 0101 \rightarrow 0011$$

(a) Right rotation for $n = 4$.

$$0101010 \rightarrow 0101001 \rightarrow 0100101 \rightarrow 0010101$$

(b) Right rotation for $n = 7$.

Figure 5: Mutual transfers for $n = 4$ and $n = 7$.

Case 2: When $n = 3m+1$ and $k = m+1$, we have to examine the following three subcases separately: 1) two vertices dominate four vertices and the remaining $k-2$ $(= m-1)$ vertices dominate $3m-3$ vertices; 2) three vertices dominate seven vertices and the remaining $k-3$ vertices dominate $3m-6$ vertices; and 3) four vertices dominate ten vertices and the remaining $k-2$ vertices dominate $3m-9$ vertices. When $n \geq 7$, any dominating set in the first subcase can be transferred to that in the second subcase (i.e., by changing subsequence `0110 010` to `0101010`), and when $n \geq 10$, any dominating set in the second subcase can be transferred to that in the third subcase (i.e., by changing subsequence `0101010 010` to `01010 01010`). In addition, mutual transfers for $n = 4$ and 7 can be easily verified by hand (see Figure 5). Thus, in the following, we merely consider dominating sets in the third subcase.

Any dominating set in the third subcase contains two copies of `01010` in it (the remaining part can be divided into $k-2$ copies of `010`). Let us consider a contraction of subsequence `101` to a "super vertex" of label 1, and mark it to distinguish it from the other label 1 vertices. Since this contraction reduces a dominating set with $m+1$ vertices on R_n to a dominating set with m vertices on $R_{3(m-1)+2}$, we can apply the same argument to Case 1 to complete the proof for this case, provided that the following two operations are available on the original ring R_n:

- an exchange of super vertex of label 1 with a neighbor of label 0, and
- an exchange of super vertex of label 1 with a neighbor of label 1.

In fact, the first operation is realized in two steps as:

$$\texttt{010}\underline{\texttt{101}}\texttt{0010} \rightarrow \texttt{0101001010} \rightarrow \texttt{0100}\underline{\texttt{101}}\texttt{010}$$

and the second operation is realized in one step as:

$$\texttt{010}\underline{\texttt{101}}\texttt{0010} \rightarrow \texttt{0101001}\underline{\texttt{01}}\texttt{0}.$$

101010 → 101001 → 100101 → 010101

(a) Right rotation for $n = 6$.

010101010 → 010101001 → 010100101
→ 010010101 → 001010101

(b) Right rotation for $n = 9$.

Figure 6: Mutual transfers for $n = 6$ and $n = 9$.

Case 3: When $n = 3m$ and $k = m + 1$, we have to examine the following four subcases separately: 1) three vertices dominate six vertices and the remaining $k - 3$ $(= m - 2)$ vertices dominate $3m - 6$ vertices; 2) four vertices dominate nine vertices and the remaining $k - 4$ vertices dominate $3m - 9$ vertices; 3) five vertices dominate 12 vertices and the remaining $k - 5$ vertices dominate $3m - 12$ vertices; and 4) six vertices dominate 15 vertices and the remaining $k - 6$ vertices dominate $3m - 15$ vertices. Note that any dominating set in the third and the fourth subcases must contain subsequence `01010`.

When $n \geq 9$, any dominating set in the first subcase can be transferred to that in the second subcase (i.e., by changing subsequence `011010 010` to `010101010`), and when $n \geq 12$, any dominating set in the second subcase can be transferred to that in the third subcase (i.e., by changing subsequence `010101010 010` to `0101010 01010`). In addition, mutual transfers for $n = 6$ and 9 can easily be verified by hand (see Figure 6). Since any dominating set in the third and the fourth subcases contains subsequence `01010`, by using the same technique to Case 2, we can reduce a dominating set for R_n with $m + 1$ vertices to a dominating set for $R_{3(m-1)+1}$ with m vertices. Hence the theorem follows. Q.E.D.

4.3.2 Preprocessing for Reduction

Let $G = (V, E)$ be a Hamiltonian graph with n vertices, and R be a Hamiltonian cycle in it. In what follows, edges contained in R will be referred to as *ring edges* and the other edges in G will be referred to as *chord edges*. Let $S \subseteq V$ be a dominating set for G with at least $\lceil (n + 1)/3 \rceil$ vertices. In the following, we will transfer S to a dominating configuration for R by consecutively removing chord edges and by moving the role of dominating vertices accordingly.

In the first step of the transfer, we apply the following rule until it could not be applied to the resultant graph:

Rule 1: If the removal of a chord edge does not violate the condition of domination for its end vertices, then remove it.

Let G' be the resultant graph. Note that S is a dominating set for G', and graph G' contains at most $n - \lceil (n + 1)/3 \rceil - 2$ $(= |V - S| - 2)$ chord edges, since there are at most $|V - S|$ vertices to be dominated by vertices in S, and at least two of them have already been dominated via ring edges. In addition, for any chord edge in G', exactly one of the end vertices must be a member of S and the other vertex must be connected with exactly one chord edge (otherwise, Rule 1 can be applied to remove a chord edge).

As the next step, we consider a subgraph G'' of G' that is obtained by removing all ring edges incident to the vertices dominated via chord edges. By construction, G'' is a forest of trees such that every leaf is a member in $V - S$ and every vertex with degree more than two is a member in S (in what follows, we call such a vertex "branch" vertex). Since $|S| \geq \lceil (n + 1)/3 \rceil$ is assumed, in at lease one of the resultant trees, the number of dominating vertices exceeds one third of the number of vertices. Let T be one of such trees and S_T $(\subseteq S)$ be the set of dominating vertices contained in T.

In the following, we will show that in graph G'', S_T can be transferred to a dominating configuration for T in which at least one leaf is a dominating one, by using the fact that $|S_T|$ is greater than one third of the number of vertices in T. Note that the proof of the above claim completes the proof of the theorem since it implies that at least one chord edge can always be removed from G' and the same argument holds for the resultant graph as long as there remains a chord edge in it; i.e., we could transfer the given configuration S for G to a dominating configuration for R (note that in the sequence of reductions, we will have to replace G'' with a new subgraph after removing a chord edge from G').

4.3.3 Transfer of S_T

Tree T contains exactly two leaf vertices dominated via ring edges. Let $u_1, u_2, \ldots, u_m$ be the sequence of vertices on the path connecting those two leaf vertices, i.e., u_1 and u_m are vertices dominated via ring edges and are connected with vertices dominated via chord edges in G'.

If T contains no branch vertices, i.e., it is a linear path, then the claim obviously holds since we can immediately transfer S_T to a configuration in which either u_1 or u_m is a dominating vertex. Thus the following remark holds.

Remark 1 *We may assume that T contains at least one branch.*

Let u_i be the first branch in T; i.e., vertices $u_1, u_2, \ldots, u_{i-1}$ form a linear path connecting to u_i. Note that $u_i \in S$. Here, we may assume $i = 2$, without loss of generality, by the following two reasons:

- If $i \neq 3j+2$ for any $j \geq 0$, then we can transfer S_T in such a way that u_1 is a dominating vertex.
- If $i = 3j+2$ for $j \geq 1$, then we could reduce T to a smaller tree by removing vertices $u_1, u_2, \ldots, u_{i-2}$ without violating the condition on the ratio of dominating vertices, since those $i-2$ vertices should be dominated by $(i-2)/3$ vertices.

Remark 2 *We may assume that vertex u_2 is the first branch in T.*

In addition, if T contains exactly one branch vertex, by the same reason to above, 1) we can transfer S_T to a configuration in which u_m is a dominating vertex, or 2) we can reduce T to a star-shaped tree with at least four vertices centered at u_2 that is dominated by at least two vertices, i.e., we can transfer S_T to a configuration in which at least one leaf vertex is a dominating one. Thus in the following, we assume T contains at least two branch vertices. Let u_j be the next branch to u_2.

Remark 3 *We may assume that T contains its second branch u_j ($\neq u_2$).*

In the following, we show that for any $j \geq 3$, we can reduce T to a smaller tree or we can transfer S_T to a configuration with a dominating leaf, by examining the following four cases separately:

Case 1 (When $j = 3$): If u_3 is connected with two or more leaves, then we can reduce T by cutting edge $\{u_2, u_3\}$, and could apply the same argument to the remaining tree containing u_3, since u_2 dominates three vertices including itself. If u_3 is connected with exactly one leaf, on the other hand, we may consider the following two subcases separately: 1) if u_4 is commonly dominated by other vertex (i.e., u_4 or u_5), then u_3 can move the role of dominating vertex to its neighboring leaf; and 2) if u_4 is privately dominated by u_3 (i.e., $u_5 \notin S_T$), then we could reduce T by cutting edge $\{u_4, u_5\}$ and could apply the same argument to the remaining tree containing u_5, since vertices u_2 and u_3 dominate at least six vertices and the domination of u_4 by u_5 in T allows u_3 to move the role of dominating vertex to a leaf.

Case 2 (When $j = 4$ or 5): We may assume that no vertices from u_3 to u_{j-1} are contained in S_T, since otherwise, it could move the role of dominating vertex to any leaf vertex in T. Then, by a similar reason to Case 1, we can reduce T by cutting edge $\{u_2, u_3\}$, and could apply the same argument to the remaining tree containing u_3.

Case 3 (When $j = 3k$ or $3k+1$ for $k \geq 2$): Since $\lceil \frac{j-2}{3} \rceil = k$, the path connecting u_1 and u_{j-2} must be dominated by at least k vertices in S_T (including u_2, of course). In the following, we assume that the number of such dominating vertices is exactly equal to k, since otherwise, we could move the role of dominating vertex to any leaf in T. In addition, since $j - 2 \not\equiv 0 \pmod 3$, we may assume $u_{j-1} \in S_T$, without loss of generality. Let T' be the tree containing u_2 that is obtained by removing edge $\{u_{j-1}, u_j\}$ from T. By the above assumptions, in tree T', at least j vertices are dominated by $k = \lfloor \frac{j}{3} \rfloor$ vertices in S_T. Thus we could apply the same argument to Case 1 by identifying u_{j-1} with vertex u_2 in the proof of Case 1.

Case 4 (When $j = 3k+2$ for $k \geq 2$): By the same reason to Case 3, we may assume that the path connecting u_1 and u_{j-2} is dominated by exactly k $(= (j-2)/3)$ vertices. Let T'' be the tree containing u_2 that is obtained by removing edge $\{u_{j-2}, u_{j-1}\}$ from T. By the above assumptions, in tree T'', at least $j-1$ vertices are dominated by $k = (j-2)/3$ vertices in S_T. Thus we could apply the same argument to Case 2 by identifying u_{j-3} with vertex u_2 in the proof of the case for $j = 5$.

Hence the theorem follows.

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